

Contents

Contents	i
List of Figures	ii
List of Tables	ii
3 Systems of Particles	1
3.1 Work-energy theorem	1
3.1.1 Conservative and nonconservative forces	2
3.1.2 Integrating $\mathbf{F} = -\nabla U$	4
3.2 Conservative forces in many-particle systems	5
3.2.1 Kinetic and potential energies	5
3.2.2 Linear and angular momentum	6
3.3 Scaling of Solutions for Homogeneous Potentials	7
3.3.1 Euler's theorem for homogeneous functions	7
3.3.2 Scaled equations of motion	8
3.4 Appendix : Curvilinear Orthogonal Coordinates	9
3.4.1 Example : spherical coordinates	10

3.4.2	Vector calculus : grad, div, curl	11
-------	---	----

List of Figures

3.1	Two paths joining points A and B	2
3.2	Volume element Ω for computing divergences	11

List of Tables

Chapter 3

Systems of Particles

3.1 Work-energy theorem

Consider a system of many particles, with positions $\mathbf{r}_i$ and velocities $\dot{\mathbf{r}}_i$. The kinetic energy of this system is

$$T = \sum_i T_i = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \quad . \quad (3.1)$$

Now let's consider how the kinetic energy of the system changes in time. Assuming each m_i is time-independent, we have

$$\frac{dT_i}{dt} = m_i \dot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}_i \quad . \quad (3.2)$$

Here, we've used the relation

$$\frac{d}{dt} (A^2) = 2 A \cdot \frac{dA}{dt} \quad . \quad (3.3)$$

We now invoke Newton's 2nd Law, $m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i$, to write eqn. 3.2 as $\dot{T}_i = \mathbf{F}_i \cdot \dot{\mathbf{r}}_i$. We integrate this equation from time t_A to t_B :

$$T_i^B - T_i^A = \int_{t_A}^{t_B} dt \frac{dT_i}{dt} = \int_{t_A}^{t_B} dt \mathbf{F}_i \cdot \dot{\mathbf{r}}_i \equiv \sum_i W_i^{A \rightarrow B} \quad , \quad (3.4)$$

where $W_i^{A \rightarrow B}$ is the total *work done* on particle i during its motion from state A to state B . Clearly the total kinetic energy is $T = \sum_i T_i$ and the total work done on all particles is $W^{A \rightarrow B} = \sum_i W_i^{A \rightarrow B}$. Eqn. 3.4 is known as the *work-energy theorem*. It says that *In the evolution of a mechanical system, the change in total kinetic energy is equal to the total work done*: $T^B - T^A = W^{A \rightarrow B}$.

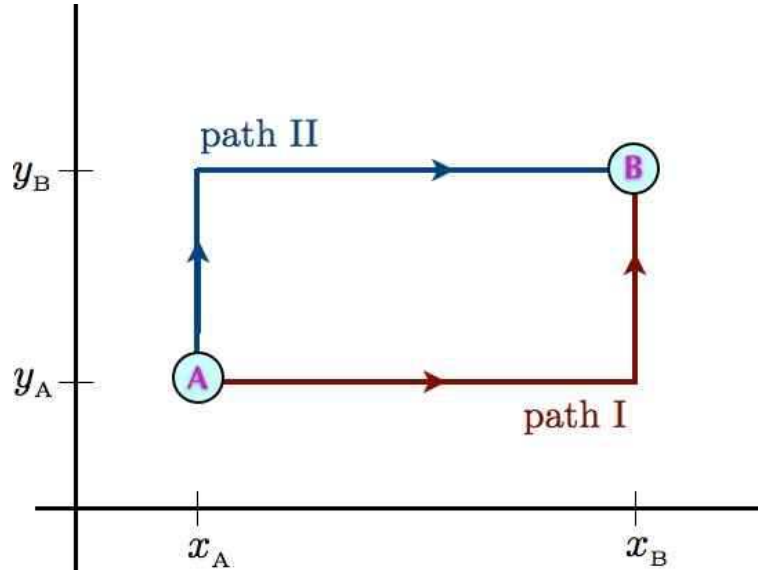


Figure 3.1: Two paths joining points A and B.

3.1.1 Conservative and nonconservative forces

For the sake of simplicity, consider a single particle with kinetic energy $T = \frac{1}{2}m\dot{\mathbf{r}}^2$. The work done on the particle during its mechanical evolution is

$$W^{A \rightarrow B} = \int_{t_A}^{t_B} dt \mathbf{F} \cdot \mathbf{v} \quad , \quad (3.5)$$

where $\mathbf{v} = \dot{\mathbf{r}}$. This is the most general expression for the work done. If the force $\mathbf{F}$ depends only on the particle's position $\mathbf{r}$, we may write $d\mathbf{r} = \mathbf{v} dt$, and then

$$W^{A \rightarrow B} = \int_{\mathbf{r}_A}^{\mathbf{r}_B} d\mathbf{r} \cdot \mathbf{F}(\mathbf{r}) \quad . \quad (3.6)$$

Consider now the force

$$\mathbf{F}(\mathbf{r}) = K_1 y \hat{\mathbf{x}} + K_2 x \hat{\mathbf{y}} \quad , \quad (3.7)$$

where $K_{1,2}$ are constants. Let's evaluate the work done along each of the two paths in fig. 3.1:

$$\begin{aligned} W^{(I)} &= K_1 \int_{x_A}^{x_B} dx y_A + K_2 \int_{y_A}^{y_B} dy x_B = K_1 y_A (x_B - x_A) + K_2 x_B (y_B - y_A) \\ W^{(II)} &= K_1 \int_{x_A}^{x_B} dx y_B + K_2 \int_{y_A}^{y_B} dy x_A = K_1 y_B (x_B - x_A) + K_2 x_A (y_B - y_A) \quad . \end{aligned} \quad (3.8)$$

Note that in general $W^{(I)} \neq W^{(II)}$. Thus, if we start at point A, the kinetic energy at point B will depend on the path taken, since the work done is path-dependent.

The difference between the work done along the two paths is

$$W^{(I-)} - W^{(II)} = (K_2 - K_1)(x_B - x_A)(y_B - y_A) \quad . \quad (3.9)$$

Thus, we see that if $K_1 = K_2$, the work is the same for the two paths. In fact, if $K_1 = K_2$, the work would be path-independent, and would depend only on the endpoints. This is true for *any* path, and not just piecewise linear paths of the type depicted in fig. 3.1. The reason for this is Stokes' theorem:

$$\oint_{\partial\mathcal{C}} d\boldsymbol{\ell} \cdot \mathbf{F} = \int_{\mathcal{C}} dS \, \hat{\mathbf{n}} \cdot \nabla \times \mathbf{F} \quad . \quad (3.10)$$

Here, $\mathcal{C}$ is a connected region in three-dimensional space, $\partial\mathcal{C}$ is mathematical notation for the boundary of $\mathcal{C}$, which is a closed path¹, dS is the scalar differential area element, $\hat{\mathbf{n}}$ is the unit normal to that differential area element, and $\nabla \times \mathbf{F}$ is the curl of $\mathbf{F}$:

$$\begin{aligned} \nabla \times \mathbf{F} &= \det \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{pmatrix} \\ &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{\mathbf{z}} \quad . \end{aligned} \quad (3.11)$$

For the force under consideration, $\mathbf{F}(\mathbf{r}) = K_1 y \hat{\mathbf{x}} + K_2 x \hat{\mathbf{y}}$, the curl is

$$\nabla \times \mathbf{F} = (K_2 - K_1) \hat{\mathbf{z}} \quad , \quad (3.12)$$

which is a constant. The RHS of eqn. 3.10 is then simply proportional to the area enclosed by $\mathcal{C}$. When we compute the work difference in eqn. 3.9, we evaluate the integral $\oint_{\mathcal{C}} d\boldsymbol{\ell} \cdot \mathbf{F}$ along the path $\gamma_{II}^{-1} \circ \gamma_I$, which is to say path I followed by the inverse of path II. In this case, $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ and the integral of $\hat{\mathbf{n}} \cdot \nabla \times \mathbf{F}$ over the rectangle $\mathcal{C}$ is given by the RHS of eqn. 3.9.

When $\nabla \times \mathbf{F} = 0$ everywhere in space, we can always write $\mathbf{F} = -\nabla U$, where $U(\mathbf{r})$ is the *potential energy*. Such forces are called *conservative forces* because the *total energy* of the system, $E = T + U$, is then conserved during its motion. We can see this by evaluating the work done,

$$W^{A \rightarrow B} = \int_{r_A}^{r_B} d\mathbf{r} \cdot \mathbf{F}(\mathbf{r}) = - \int_{r_A}^{r_B} d\mathbf{r} \cdot \nabla U = U(\mathbf{r}_A) - U(\mathbf{r}_B) \quad . \quad (3.13)$$

The work-energy theorem then gives

$$T^B - T^A = U(\mathbf{r}_A) - U(\mathbf{r}_B) \quad , \quad (3.14)$$

which says

$$E^B = T^B + U(\mathbf{r}_B) = T^A + U(\mathbf{r}_A) = E^A \quad . \quad (3.15)$$

Thus, the total energy $E = T + U$ is conserved.

¹If $\mathcal{C}$ is multiply connected, then $\partial\mathcal{C}$ is a set of closed paths. For example, if $\mathcal{C}$ is an annulus, $\partial\mathcal{C}$ is two circles, corresponding to the inner and outer boundaries of the annulus.

3.1.2 Integrating $\mathbf{F} = -\nabla U$

If $\nabla \times \mathbf{F} = 0$, we can compute $U(\mathbf{r})$ by integrating, *viz.*

$$U(\mathbf{r}) = U(\mathbf{0}) - \int_{\mathbf{0}}^{\mathbf{r}} d\mathbf{r}' \cdot \mathbf{F}(\mathbf{r}') \quad . \quad (3.16)$$

The integral does not depend on the path chosen connecting $\mathbf{0}$ and $\mathbf{r}$. For example, we can take

$$U(x, y, z) = U(0, 0, 0) - \int_{(0,0,0)}^{(x,0,0)} dx' F_x(x', 0, 0) - \int_{(x,0,0)}^{(x,y,0)} dy' F_y(x, y', 0) - \int_{(x,y,0)}^{(x,y,z)} dz' F_z(x, y, z') \quad . \quad (3.17)$$

The constant $U(0, 0, 0)$ is arbitrary and impossible to determine from $\mathbf{F}$ alone.

As an example, consider the force

$$\mathbf{F}(\mathbf{r}) = -ky \hat{\mathbf{x}} - kx \hat{\mathbf{y}} - 4bz^3 \hat{\mathbf{z}} \quad , \quad (3.18)$$

where k and b are constants. We have

$$\begin{aligned} (\nabla \times \mathbf{F})_x &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) = 0 \\ (\nabla \times \mathbf{F})_y &= \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) = 0 \\ (\nabla \times \mathbf{F})_z &= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) = 0 \quad , \end{aligned} \quad (3.19)$$

so $\nabla \times \mathbf{F} = 0$ and $\mathbf{F}$ must be expressible as $\mathbf{F} = -\nabla U$. Integrating using eqn. 3.17, we have

$$\begin{aligned} U(x, y, z) &= U(0, 0, 0) + \int_{(0,0,0)}^{(x,0,0)} dx' k \cdot 0 + \int_{(x,0,0)}^{(x,y,0)} dy' kxy' + \int_{(x,y,0)}^{(x,y,z)} dz' 4bz'^3 \\ &= U(0, 0, 0) + kxy + bz^4 \quad . \end{aligned} \quad (3.20)$$

Another approach is to integrate the partial differential equation $\nabla U = -\mathbf{F}$. This is in fact three equations, and we shall need all of them to obtain the correct answer. We start with the $\hat{\mathbf{x}}$ -component,

$$\frac{\partial U}{\partial x} = ky \quad . \quad (3.21)$$

Integrating, we obtain

$$U(x, y, z) = kxy + f(y, z) \quad , \quad (3.22)$$

where $f(y, z)$ is at this point an *arbitrary function* of y and z . The important thing is that it has no x -dependence, so $\partial f/\partial x = 0$. Next, we have

$$\frac{\partial U}{\partial y} = kx \implies U(x, y, z) = kxy + g(x, z) \quad . \quad (3.23)$$

Finally, the z -component integrates to yield

$$\frac{\partial U}{\partial z} = 4bz^3 \implies U(x, y, z) = bz^4 + h(x, y) \quad . \quad (3.24)$$

We now equate the first two expressions:

$$kxy + f(y, z) = kxy + g(x, z) \quad . \quad (3.25)$$

Subtracting kxy from each side, we obtain the equation $f(y, z) = g(x, z)$. Since the LHS is independent of x and the RHS is independent of y , we must have

$$f(y, z) = g(x, z) = q(z) \quad , \quad (3.26)$$

where $q(z)$ is some unknown function of z . But now we invoke the final equation, to obtain

$$bz^4 + h(x, y) = kxy + q(z) \quad . \quad (3.27)$$

The only possible solution is $h(x, y) = C + kxy$ and $q(z) = C + bz^4$, where C is a constant. Therefore,

$$U(x, y, z) = C + kxy + bz^4 \quad . \quad (3.28)$$

Note that it would be *very wrong* to integrate $\partial U/\partial x = ky$ and obtain $U(x, y, z) = kxy + C'$, where C' is a constant. As we've seen, the 'constant of integration' we obtain upon integrating this first order PDE is in fact a *function* of y and z . The fact that $f(y, z)$ carries no explicit x dependence means that $\partial f/\partial x = 0$, so by construction $U = kxy + f(y, z)$ is a solution to the PDE $\partial U/\partial x = ky$, for any arbitrary function $f(y, z)$.

3.2 Conservative forces in many-particle systems

3.2.1 Kinetic and potential energies

The kinetic and potential energies are given by

$$\begin{aligned} T &= \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \\ U &= \sum_i V(\mathbf{r}_i) + \sum_{i < j} v(|\mathbf{r}_i - \mathbf{r}_j|) \quad . \end{aligned} \quad (3.29)$$

Here, $V(\mathbf{r})$ is the *external* (or one-body) potential, and $v(\mathbf{r} - \mathbf{r}')$ is the *interparticle* potential, which we assume to be central, depending only on the distance between any pair of particles. The equations of motion are

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i^{(\text{ext})} + \mathbf{F}_i^{(\text{int})} \quad , \quad (3.30)$$

with

$$\begin{aligned} \mathbf{F}_i^{(\text{ext})} &= -\frac{\partial V(\mathbf{r}_i)}{\partial \mathbf{r}_i} \\ \mathbf{F}_i^{(\text{int})} &= -\sum_j \frac{\partial v(|\mathbf{r}_i - \mathbf{r}_j|)}{\partial \mathbf{r}_i} \equiv \sum_j \mathbf{F}_{ij}^{(\text{int})} . \end{aligned} \quad (3.31)$$

Here, $\mathbf{F}_{ij}^{(\text{int})}$ is the force exerted on particle i by particle j :

$$\mathbf{F}_{ij}^{(\text{int})} = -\frac{\partial v(|\mathbf{r}_i - \mathbf{r}_j|)}{\partial \mathbf{r}_i} = -\frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|} v'(|\mathbf{r}_i - \mathbf{r}_j|) . \quad (3.32)$$

Note that $\mathbf{F}_{ij}^{(\text{int})} = -\mathbf{F}_{ji}^{(\text{int})}$, otherwise known as Newton's Third Law. It is convenient to abbreviate $\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j$ in which case we may write the interparticle force as

$$\mathbf{F}_{ij}^{(\text{int})} = -\hat{\mathbf{r}}_{ij} v'(\mathbf{r}_{ij}) . \quad (3.33)$$

3.2.2 Linear and angular momentum

Consider now the total momentum of the system, $\mathbf{P} = \sum_i \mathbf{p}_i$. Its rate of change is

$$\frac{d\mathbf{P}}{dt} = \sum_i \dot{\mathbf{p}}_i = \sum_i \mathbf{F}_i^{(\text{ext})} + \overbrace{\sum_{i \neq j} \mathbf{F}_{ij}^{(\text{int})}}^{\mathbf{F}_{ij}^{(\text{int})} + \mathbf{F}_{ji}^{(\text{int})} = 0} = \mathbf{F}_{\text{tot}}^{(\text{ext})} , \quad (3.34)$$

since the sum over all internal forces cancels as a result of Newton's Third Law. We write

$$\begin{aligned} \mathbf{P} &= \sum_i m_i \dot{\mathbf{r}}_i = M \dot{\mathbf{R}} \\ M &= \sum_i m_i \quad (\text{total mass}) \\ \mathbf{R} &= \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} \quad (\text{center-of-mass}) . \end{aligned} \quad (3.35)$$

Next, consider the total angular momentum,

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i . \quad (3.36)$$

The rate of change of $\mathbf{L}$ is then

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \sum_i \{m_i \dot{\mathbf{r}}_i \times \dot{\mathbf{r}}_i + m_i \mathbf{r}_i \times \ddot{\mathbf{r}}_i\} \\ &= \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(\text{ext})} + \sum_{i \neq j} \mathbf{r}_i \times \mathbf{F}_{ij}^{(\text{int})} \\ &= \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(\text{ext})} + \overbrace{\frac{1}{2} \sum_{i \neq j} (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}^{(\text{int})}}^{\mathbf{r}_{ij} \times \mathbf{F}_{ij}^{(\text{int})} = 0} = \mathbf{N}_{\text{tot}}^{(\text{ext})} . \end{aligned} \quad (3.37)$$

Finally, it is useful to establish the result

$$T = \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}_i^2 = \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \sum_i m_i (\dot{\mathbf{r}}_i - \dot{\mathbf{R}})^2 \quad , \quad (3.38)$$

which says that the kinetic energy may be written as a sum of two terms, those being the kinetic energy of the center-of-mass motion, and the kinetic energy of the particles relative to the center-of-mass.

Recall the “work-energy theorem” for conservative systems,

$$\begin{aligned} 0 &= \int_{\text{initial}}^{\text{final}} dE = \int_{\text{initial}}^{\text{final}} dT + \int_{\text{initial}}^{\text{final}} dU \\ &= T^{\text{B}} - T^{\text{A}} - \sum_i \int d\mathbf{r}_i \cdot \mathbf{F}_i \quad , \end{aligned} \quad (3.39)$$

which is to say

$$\Delta T = T^{\text{B}} - T^{\text{A}} = \sum_i \int d\mathbf{r}_i \cdot \mathbf{F}_i = -\Delta U \quad . \quad (3.40)$$

In other words, the total energy $E = T + U$ is conserved:

$$E = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 + \sum_i V(\mathbf{r}_i) + \sum_{i < j} v(|\mathbf{r}_i - \mathbf{r}_j|) \quad . \quad (3.41)$$

Note that for continuous systems, we replace sums by integrals over a mass distribution, *viz.*

$$\sum_i m_i \phi(\mathbf{r}_i) \longrightarrow \int d^3r \rho(\mathbf{r}) \phi(\mathbf{r}) \quad , \quad (3.42)$$

where $\rho(\mathbf{r})$ is the mass density, and $\phi(\mathbf{r})$ is any function.

3.3 Scaling of Solutions for Homogeneous Potentials

3.3.1 Euler’s theorem for homogeneous functions

In certain cases of interest, the potential is a homogeneous function of the coordinates. This means

$$U(\lambda \mathbf{r}_1, \dots, \lambda \mathbf{r}_N) = \lambda^k U(\mathbf{r}_1, \dots, \mathbf{r}_N) \quad . \quad (3.43)$$

Here, k is the *degree of homogeneity* of U . Familiar examples include gravity,

$$U(\mathbf{r}_1, \dots, \mathbf{r}_N) = -G \sum_{i < j} \frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|} \quad ; \quad k = -1 \quad , \quad (3.44)$$

and the harmonic oscillator,

$$U(q_1, \dots, q_n) = \frac{1}{2} \sum_{\sigma, \sigma'} V_{\sigma\sigma'} q_\sigma q_{\sigma'} \quad ; \quad k = +2 \quad . \quad (3.45)$$

The sum of two homogeneous functions is itself homogeneous only if the component functions themselves are of the same degree of homogeneity. Homogeneous functions obey a special result known as *Euler's Theorem*, which we now prove. Suppose a multivariable function $H(x_1, \dots, x_n)$ is homogeneous:

$$H(\lambda x_1, \dots, \lambda x_n) = \lambda^k H(x_1, \dots, x_n) \quad . \quad (3.46)$$

Then

$$\left. \frac{d}{d\lambda} \right|_{\lambda=1} H(\lambda x_1, \dots, \lambda x_n) = \sum_{i=1}^n x_i \frac{\partial H}{\partial x_i} = kH \quad . \quad (3.47)$$

3.3.2 Scaled equations of motion

Now suppose the we rescale distances and times, defining

$$\mathbf{r}_i = \alpha \tilde{\mathbf{r}}_i \quad , \quad t = \beta \tilde{t} \quad . \quad (3.48)$$

Then

$$\frac{d\mathbf{r}_i}{dt} = \frac{\alpha}{\beta} \frac{d\tilde{\mathbf{r}}_i}{d\tilde{t}} \quad , \quad \frac{d^2\mathbf{r}_i}{dt^2} = \frac{\alpha}{\beta^2} \frac{d^2\tilde{\mathbf{r}}_i}{d\tilde{t}^2} \quad . \quad (3.49)$$

The force $\mathbf{F}_i$ is given by

$$\begin{aligned} \mathbf{F}_i &= -\frac{\partial}{\partial \mathbf{r}_i} U(\mathbf{r}_1, \dots, \mathbf{r}_N) \\ &= -\frac{\partial}{\partial (\alpha \tilde{\mathbf{r}}_i)} \alpha^k U(\tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_N) \equiv \alpha^{k-1} \tilde{\mathbf{F}}_i \quad , \end{aligned} \quad (3.50)$$

where $\tilde{\mathbf{F}}_i = \partial U(\tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_N) / \partial \tilde{\mathbf{r}}_i$. Thus, Newton's 2nd Law says

$$\frac{\alpha}{\beta^2} m_i \frac{d^2\tilde{\mathbf{r}}_i}{d\tilde{t}^2} = \alpha^{k-1} \tilde{\mathbf{F}}_i \quad . \quad (3.51)$$

If we choose β such that

$$\frac{\alpha}{\beta^2} = \alpha^{k-1} \quad \Rightarrow \quad \beta = \alpha^{1-\frac{1}{2}k} \quad , \quad (3.52)$$

then the equation of motion is invariant under the rescaling transformation, *i.e.*

$$m_i \frac{d^2\tilde{\mathbf{r}}_i}{d\tilde{t}^2} = \tilde{\mathbf{F}}_i \quad . \quad (3.53)$$

This means that if $\{\mathbf{r}_i(t)\}$ is a solution to the equations of motion, then so is $\{\alpha \mathbf{r}_i(\beta t)\}$. This gives us an entire one-parameter family of solutions, for all real positive α . with $\beta = \alpha^{1-\frac{1}{2}k}$.

We see that if $\mathbf{r}_i(t)$ is periodic with period T , then $\mathbf{r}_i(t; \alpha)$ is periodic with period $T' = \alpha^{1-\frac{1}{2}k} T$. Furthermore, if L is a length scale associated with an orbit $\mathbf{r}_i(t)$, such as the distance of closest approach, then we have the following relation between the ratios of the time and length scales:

$$\left(\frac{T'}{T}\right) = \left(\frac{L'}{L}\right)^{1-\frac{1}{2}k} . \quad (3.54)$$

Velocities, energies and angular momenta scale accordingly Thus

$$[v] = \frac{L}{T} \Rightarrow \frac{v'}{v} = \frac{L'}{L} \bigg/ \frac{T'}{T} = \alpha^{\frac{1}{2}k} \quad (3.55)$$

and

$$[E] = \frac{ML^2}{T^2} \Rightarrow \frac{E'}{E} = \left(\frac{L'}{L}\right)^2 \bigg/ \left(\frac{T'}{T}\right)^2 = \alpha^k \quad (3.56)$$

and

$$[L] = \frac{ML^2}{T} \Rightarrow \frac{|L'|}{|L|} = \left(\frac{L'}{L}\right)^2 \bigg/ \frac{T'}{T} = \alpha^{(1+\frac{1}{2}k)} . \quad (3.57)$$

As examples, consider:

(i) *Harmonic Oscillator* : Here $k = 2$ and therefore

$$q_\sigma(t) \longrightarrow q_\sigma(t; \alpha) = \alpha q_\sigma(t) . \quad (3.58)$$

Thus, rescaling lengths alone gives another solution.

(ii) *Kepler Problem* : This is gravity, for which $k = -1$. Thus,

$$\mathbf{r}(t) \longrightarrow \mathbf{r}(t; \alpha) = \alpha \mathbf{r}(\alpha^{-3/2} t) . \quad (3.59)$$

Thus, $r^3 \propto t^2$, i.e.

$$\left(\frac{L'}{L}\right)^3 = \left(\frac{T'}{T}\right)^2 , \quad (3.60)$$

also known as Kepler's Third Law.

3.4 Appendix : Curvilinear Orthogonal Coordinates

The standard cartesian coordinates are $\{x_1, \dots, x_d\}$, where d is the dimension of space. Consider a different set of coordinates, $\{q_1, \dots, q_d\}$, which are related to the original coordinates x_μ via the d equations

$$q_\mu = q_\mu(x_1, \dots, x_d) . \quad (3.61)$$

In general these are nonlinear equations.

Let $\hat{e}_i^0 = \hat{x}_i$ be the Cartesian set of orthonormal unit vectors, and define $\hat{e}_\mu$ to be the unit vector perpendicular to the surface $dq_\mu = 0$. A differential change in position can now be described in both coordinate systems:

$$d\mathbf{s} = \sum_{i=1}^d \hat{e}_i^0 dx_i = \sum_{\mu=1}^d \hat{e}_\mu h_\mu(q) dq_\mu \quad , \quad (3.62)$$

where each $h_\mu(q)$ is an as yet unknown function of all the components q_ν . Finding the coefficient of dq_μ then gives

$$h_\mu(q) \hat{e}_\mu = \sum_{i=1}^d \frac{\partial x_i}{\partial q_\mu} \hat{e}_i^0 \quad \Rightarrow \quad \hat{e}_\mu = \sum_{i=1}^d M_{\mu i} \hat{e}_i^0 \quad , \quad (3.63)$$

where

$$M_{\mu i}(q) = \frac{1}{h_\mu(q)} \frac{\partial x_i}{\partial q_\mu} \quad . \quad (3.64)$$

The dot product of unit vectors in the new coordinate system is then

$$\hat{e}_\mu \cdot \hat{e}_\nu = (MM^t)_{\mu\nu} = \frac{1}{h_\mu(q) h_\nu(q)} \sum_{i=1}^d \frac{\partial x_i}{\partial q_\mu} \frac{\partial x_i}{\partial q_\nu} \quad . \quad (3.65)$$

The condition that the new basis be orthonormal is then

$$\sum_{i=1}^d \frac{\partial x_i}{\partial q_\mu} \frac{\partial x_i}{\partial q_\nu} = h_\mu^2(q) \delta_{\mu\nu} \quad . \quad (3.66)$$

This gives us the relation

$$h_\mu(q) = \sqrt{\sum_{i=1}^d \left(\frac{\partial x_i}{\partial q_\mu} \right)^2} \quad . \quad (3.67)$$

Note that

$$(d\mathbf{s})^2 = \sum_{\mu=1}^d h_\mu^2(q) (dq_\mu)^2 \quad . \quad (3.68)$$

For general coordinate systems, which are not necessarily orthogonal, we have

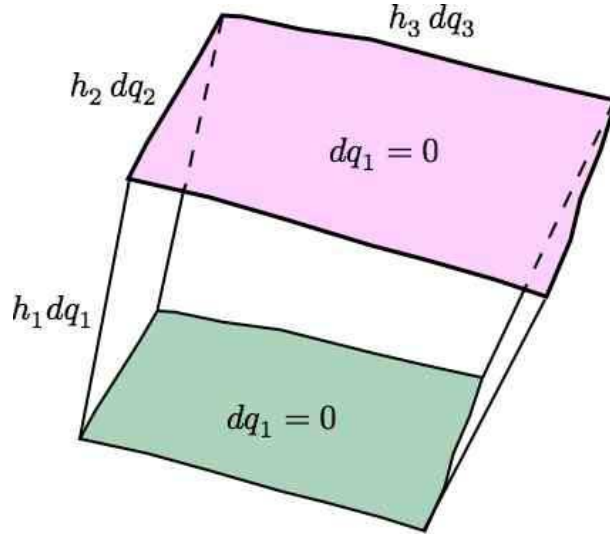
$$(d\mathbf{s})^2 = \sum_{\mu, \nu=1}^d g_{\mu\nu}(q) dq_\mu dq_\nu \quad , \quad (3.69)$$

where $g_{\mu\nu}(q)$ is a real, symmetric, positive definite matrix called the *metric tensor*.

3.4.1 Example : spherical coordinates

Consider spherical coordinates (ρ, θ, ϕ) :

$$x = \rho \sin \theta \cos \phi \quad , \quad y = \rho \sin \theta \sin \phi \quad , \quad z = \rho \cos \theta \quad . \quad (3.70)$$

Figure 3.2: Volume element Ω for computing divergences.

It is now a simple matter to derive the results

$$h_\rho^2 = 1 \quad , \quad h_\theta^2 = \rho^2 \quad , \quad h_\phi^2 = \rho^2 \sin^2 \theta \quad . \quad (3.71)$$

Thus,

$$ds = \hat{\rho} d\rho + \rho \hat{\theta} d\theta + \rho \sin \theta \hat{\phi} d\phi \quad . \quad (3.72)$$

3.4.2 Vector calculus : grad, div, curl

Here we restrict our attention to $d = 3$. The gradient ∇U of a function $U(q)$ is defined by

$$\begin{aligned} dU &= \frac{\partial U}{\partial q_1} dq_1 + \frac{\partial U}{\partial q_2} dq_2 + \frac{\partial U}{\partial q_3} dq_3 \\ &\equiv \nabla U \cdot ds \quad . \end{aligned} \quad (3.73)$$

Thus,

$$\nabla = \frac{\hat{e}_1}{h_1(q)} \frac{\partial}{\partial q_1} + \frac{\hat{e}_2}{h_2(q)} \frac{\partial}{\partial q_2} + \frac{\hat{e}_3}{h_3(q)} \frac{\partial}{\partial q_3} \quad . \quad (3.74)$$

For the divergence, we use the divergence theorem, and we appeal to fig. 3.2:

$$\int_{\Omega} dV \nabla \cdot \mathbf{A} = \int_{\partial\Omega} dS \hat{\mathbf{n}} \cdot \mathbf{A} \quad , \quad (3.75)$$

where Ω is a region of three-dimensional space and $\partial\Omega$ is its closed two-dimensional boundary. The LHS of this equation is

$$\text{LHS} = \nabla \cdot \mathbf{A} \cdot (h_1 dq_1) (h_2 dq_2) (h_3 dq_3) \quad . \quad (3.76)$$

The RHS is

$$\begin{aligned} \text{RHS} &= A_1 h_2 h_3 \Big|_{q_1}^{q_1+dq_1} dq_2 dq_3 + A_2 h_1 h_3 \Big|_{q_2}^{q_2+dq_2} dq_1 dq_3 + A_3 h_1 h_2 \Big|_{q_3}^{q_3+dq_3} dq_1 dq_2 \\ &= \left[\frac{\partial}{\partial q_1} (A_1 h_2 h_3) + \frac{\partial}{\partial q_2} (A_2 h_1 h_3) + \frac{\partial}{\partial q_3} (A_3 h_1 h_2) \right] dq_1 dq_2 dq_3 . \end{aligned} \quad (3.77)$$

We therefore conclude

$$\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (A_1 h_2 h_3) + \frac{\partial}{\partial q_2} (A_2 h_1 h_3) + \frac{\partial}{\partial q_3} (A_3 h_1 h_2) \right] . \quad (3.78)$$

To obtain the curl $\nabla \times \mathbf{A}$, we use Stokes' theorem again,

$$\int_{\Sigma} dS \hat{\mathbf{n}} \cdot \nabla \times \mathbf{A} = \oint_{\partial \Sigma} d\ell \cdot \mathbf{A} , \quad (3.79)$$

where Σ is a two-dimensional region of space and $\partial \Sigma$ is its one-dimensional boundary. Now consider a differential surface element satisfying $dq_1 = 0$, *i.e.* a rectangle of side lengths $h_2 dq_2$ and $h_3 dq_3$. The LHS of the above equation is

$$\text{LHS} = \hat{\mathbf{e}}_1 \cdot \nabla \times \mathbf{A} (h_2 dq_2) (h_3 dq_3) . \quad (3.80)$$

The RHS is

$$\begin{aligned} \text{RHS} &= A_3 h_3 \Big|_{q_2}^{q_2+dq_2} dq_3 - A_2 h_2 \Big|_{q_3}^{q_3+dq_3} dq_2 \\ &= \left[\frac{\partial}{\partial q_2} (A_3 h_3) - \frac{\partial}{\partial q_3} (A_2 h_2) \right] dq_2 dq_3 . \end{aligned} \quad (3.81)$$

Therefore

$$(\nabla \times \mathbf{A})_1 = \frac{1}{h_2 h_3} \left(\frac{\partial (h_3 A_3)}{\partial q_2} - \frac{\partial (h_2 A_2)}{\partial q_3} \right) . \quad (3.82)$$

This is one component of the full result

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \det \begin{pmatrix} h_1 \hat{\mathbf{e}}_1 & h_2 \hat{\mathbf{e}}_2 & h_3 \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{pmatrix} . \quad (3.83)$$

The Laplacian of a scalar function U is given by

$$\begin{aligned} \nabla^2 U &= \nabla \cdot \nabla U \\ &= \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial U}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial U}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial U}{\partial q_3} \right) \right\} . \end{aligned} \quad (3.84)$$

Rectangular coordinates

In *rectangular* coordinates (x, y, z) , we have

$$h_x = h_y = h_z = 1 \quad . \quad (3.85)$$

Thus

$$ds = \hat{x} dx + \hat{y} dy + \hat{z} dz \quad (3.86)$$

and the velocity squared is

$$\dot{s}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \quad . \quad (3.87)$$

The gradient is

$$\nabla U = \hat{x} \frac{\partial U}{\partial x} + \hat{y} \frac{\partial U}{\partial y} + \hat{z} \frac{\partial U}{\partial z} \quad . \quad (3.88)$$

The divergence is

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad . \quad (3.89)$$

The curl is

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{z} \quad . \quad (3.90)$$

The Laplacian is

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \quad . \quad (3.91)$$

Cylindrical coordinates

In *cylindrical* coordinates (ρ, ϕ, z) , we have

$$\hat{\rho} = \hat{x} \cos \phi + \hat{y} \sin \phi \quad , \quad \hat{x} = \hat{\rho} \cos \phi - \hat{\phi} \sin \phi \quad , \quad d\hat{\rho} = \hat{\phi} d\phi \quad (3.92)$$

and

$$\hat{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi \quad , \quad \hat{y} = \hat{\rho} \sin \phi + \hat{\phi} \cos \phi \quad , \quad d\hat{\phi} = -\hat{\rho} d\phi \quad . \quad (3.93)$$

The metric is given in terms of

$$h_\rho = 1 \quad , \quad h_\phi = \rho \quad , \quad h_z = 1 \quad . \quad (3.94)$$

Thus

$$ds = \hat{\rho} d\rho + \hat{\phi} \rho d\phi + \hat{z} dz \quad (3.95)$$

and the velocity squared is

$$\dot{s}^2 = \dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2 \quad . \quad (3.96)$$

The gradient is

$$\nabla U = \hat{\rho} \frac{\partial U}{\partial \rho} + \frac{\hat{\phi}}{\rho} \frac{\partial U}{\partial \phi} + \hat{z} \frac{\partial U}{\partial z} \quad . \quad (3.97)$$

The divergence is

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} . \quad (3.98)$$

The curl is

$$\nabla \times \mathbf{A} = \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \hat{\phi} + \left(\frac{1}{\rho} \frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{1}{\rho} \frac{\partial A_\rho}{\partial \phi} \right) \hat{z} . \quad (3.99)$$

The Laplacian is

$$\nabla^2 U = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial U}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} . \quad (3.100)$$

Spherical coordinates

In *spherical* coordinates (r, θ, ϕ) , we have

$$\begin{aligned} \hat{\mathbf{r}} &= \hat{\mathbf{x}} \sin \theta \cos \phi + \hat{\mathbf{y}} \sin \theta \sin \phi + \hat{\mathbf{z}} \cos \theta \\ \hat{\boldsymbol{\theta}} &= \hat{\mathbf{x}} \cos \theta \cos \phi + \hat{\mathbf{y}} \cos \theta \sin \phi - \hat{\mathbf{z}} \sin \theta \\ \hat{\boldsymbol{\phi}} &= -\hat{\mathbf{x}} \sin \phi + \hat{\mathbf{y}} \cos \phi , \end{aligned} \quad (3.101)$$

for which

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} , \quad \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{r}} , \quad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}} . \quad (3.102)$$

The inverse is

$$\begin{aligned} \hat{\mathbf{x}} &= \hat{\mathbf{r}} \sin \theta \cos \phi + \hat{\boldsymbol{\theta}} \cos \theta \cos \phi - \hat{\boldsymbol{\phi}} \sin \phi \\ \hat{\mathbf{y}} &= \hat{\mathbf{r}} \sin \theta \sin \phi + \hat{\boldsymbol{\theta}} \cos \theta \sin \phi + \hat{\boldsymbol{\phi}} \cos \phi \\ \hat{\mathbf{z}} &= \hat{\mathbf{r}} \cos \theta - \hat{\boldsymbol{\theta}} \sin \theta . \end{aligned} \quad (3.103)$$

The differential relations are

$$\begin{aligned} d\hat{\mathbf{r}} &= \hat{\boldsymbol{\theta}} d\theta + \sin \theta \hat{\boldsymbol{\phi}} d\phi \\ d\hat{\boldsymbol{\theta}} &= -\hat{\mathbf{r}} d\theta + \cos \theta \hat{\boldsymbol{\phi}} d\phi \\ d\hat{\boldsymbol{\phi}} &= -(\sin \theta \hat{\mathbf{r}} + \cos \theta \hat{\boldsymbol{\theta}}) d\phi \end{aligned} \quad (3.104)$$

The metric is given in terms of

$$h_r = 1 , \quad h_\theta = r , \quad h_\phi = r \sin \theta . \quad (3.105)$$

Thus

$$ds = \hat{\mathbf{r}} dr + \hat{\boldsymbol{\theta}} r d\theta + \hat{\boldsymbol{\phi}} r \sin \theta d\phi \quad (3.106)$$

and the velocity squared is

$$\dot{\mathbf{s}}^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 . \quad (3.107)$$

The gradient is

$$\nabla U = \hat{\mathbf{r}} \frac{\partial U}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r} \frac{\partial U}{\partial \theta} + \frac{\hat{\boldsymbol{\phi}}}{r \sin \theta} \frac{\partial U}{\partial \phi} \quad . \quad (3.108)$$

The divergence is

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\sin \theta A_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \quad . \quad (3.109)$$

The curl is

$$\begin{aligned} \nabla \times \mathbf{A} = & \frac{1}{r \sin \theta} \left(\frac{\partial(\sin \theta A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi} \right) \hat{\mathbf{r}} + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial(r A_\phi)}{\partial r} \right) \hat{\boldsymbol{\theta}} \\ & + \frac{1}{r} \left(\frac{\partial(r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right) \hat{\boldsymbol{\phi}} \quad . \end{aligned} \quad (3.110)$$

The Laplacian is

$$\nabla^2 U = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial U}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 U}{\partial \phi^2} \quad . \quad (3.111)$$

Kinetic energy

Note the form of the kinetic energy of a point particle:

$$\begin{aligned} T = \frac{1}{2} m \left(\frac{d\mathbf{s}}{dt} \right)^2 &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) && \text{(3D Cartesian)} \\ &= \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\phi}^2) && \text{(2D polar)} \\ &= \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2) && \text{(3D cylindrical)} \\ &= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) && \text{(3D polar)} \quad . \end{aligned} \quad (3.112)$$